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2001 J. Phys. A: Math. Gen. 34 9329

(http://iopscience.iop.org/0305-4470/34/43/315)

J. Phys. A: Math. Gen. **34** (2001) 9329–9338 PII: S0305-4470(01)21882-4

Path integrals on a manifold with group action

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Received 9 February 2001
Published 19 October 2001
Online at stacks.iop.org/JPhysA/34/9329

Abstract
A new path integral measure factorization method is proposed. By using this method the reduction procedure in Wiener path integrals for a scalar particle on a smooth compact Riemannian manifold with the given free isometric action of the compact semisimple unimodular Lie group is considered. It is shown that the path integral measure is not invariant under the reduction. The integral relation between path integrals representing the fundamental solutions of the parabolic equations on initial and reduced manifolds is derived.

PACS numbers: 03.65.Bz, 02.20.-a, 02.40.-k, 02.50.Ey, 31.15.Kb

1. Introduction

Recently, interest in the problem of path integral quantization of finite-dimensional systems with a symmetry has received renewed attention [1]. One can meet these systems in various branches of physics.

However, there is another reason why we are interested in the path integral quantization of these systems. It is supposed that new path integral quantization approaches could be applied to the infinite dynamical systems with gauge symmetries. In this connection, the finite-dimensional system which describes the motion of a scalar particle on the Riemannian manifold with the given free isometric Lie group action is especially attractive for investigations [2].

Having such an action of the group on a manifold, we can view the manifold as a local fibre space. Moreover, there arises a principal bundle structure with the connection induced in a natural way by a metric of the manifold. In [3] this connection was called the mechanical connection.

In the case of motion under the group-invariant potential, the initial dynamical system is reduced to the system given on the orbit space. It is due to this fact that we can view our system as a model system in studying the interrelation between the quantum motions of initial and reduced systems. This interrelation is the main point in the problem of quantum reduction of the constrained systems.

In this paper we will study the reduction procedure in the path integral for the motion of a scalar particle on the smooth compact Riemannian manifold on which the free isometric action of the compact unimodular semisimple Lie group is given. By the path integral reduction procedure we mean such a path integral transformation, when the initial space is changed for the reduced one.

There are a lot of papers devoted to this problem [1, 2, 4], but in spite of this, the question of the correct value of the reduction Jacobian has not received a definite answer.

Most of the papers concerning the path integral reduction problem deal with the Feynman path integrals defined by discrete approximations. In our paper we will consider the case of the Wiener path integrals, in which the integration measures are generated by the stochastic processes. The stochastic processes will be determined by solutions of the stochastic differential equations that are given on manifolds.

To define the stochastic processes (and the stochastic differential equations) on a manifold we will use the method developed by Belopolskaya and Dalecky in [6]. This method is based on a local description of stochastic processes. In the chart of the manifold the stochastic processes are given by the definite stochastic differential equations. The equations are the result of the exponential mapping from the corresponding stochastic differential equation defined on the tangent bundle over the manifold. On overlapping of the charts the local equations and their solutions transform into each other.

By using the local stochastic processes obtained after subdividing the time interval it is possible to get the directed stochastic evolution family of the manifold mappings. In the case of the compact manifold and when, in addition, some of the analytical restrictions are imposed on the linear connection (the fulfilment of this requirement will be assumed in this paper) the directed evolution family has a limit [6], which defines the global stochastic process on a manifold. A similar scheme of the stochastic process definition is valid for a vector and a principal bundle too [6].

In this paper we confine ourselves to the case when the effects coming from the nontrivial topology of the manifold are not important. As a consequence of this reason the investigation of the path integral reduction can be made in the local charts of the manifold. Afterwards, the transition to the global picture can be realized by the method developed in [6].

The main problem of the path integral reduction is the separation of integration variables. We should separate the variables associated with the group action on a manifold from variables that are projected into the base of the principal bundle. In other words, it is necessary to separate the invariant variables from the variables that are changeable under the group action. In this paper we get such a separation of variables by using the so-called 'nonlinear filtering equation' from the stochastic process theory. Using this equation, we will derive the integral relation between path integrals representing the fundamental solutions of parabolic equations defined on the initial and reduced manifold.

In the case of the 'nonzero momentum level reduction' (in terms of the constrained dynamical system theory) the path integral induced on the orbit space represents the fundamental solution of the linear parabolic system of the differential equations. In the 'zero-momentum level reduction' case, which is also considered in this paper, the path integrals on the initial and reduced manifold serve to describe the motions of the scalar particles.

The investigation performed in this paper leads to the conclusion that the path integral measure is not invariant under the reduction procedure.

2. Definitions

Let the backward Kolmogorov equation be given on a smooth compact Riemannian manifold $\mathcal{P}$:

$$\left(\frac{\partial}{\partial t_a}+\frac{1}{2}\mu^2\kappa\triangle_{\rm P}(Q_a)+\frac{1}{\mu^2\kappa m}V(Q_a)\right)\psi(Q_a,t_a)=0$$
$$\psi(Q_b,t_b)=\varphi_0(Q_b)\qquad (t_b>t_a) \tag{1}$$

$\mu^2=\frac{\hbar}{m}$, κ is a real positive parameter,

$$\triangle_{\rm P}(Q_a)=G^{-1/2}\frac{\partial}{\partial Q_a^A}G^{AB}G^{1/2}\frac{\partial}{\partial Q_a^B}$$

is the Laplace–Beltrami operator on $\mathcal{P}$, and $G=\det G_{AB}$ (the indices denoted by capital letters run from 1 to $n_{\rm P}=\dim\mathcal{P}$). If the coefficients of equation (1) and the function φ_0 all satisfy (as in [6]) the necessary smooth requirements, then the solution of equation (1) can be represented in the form

$$\psi(Q_a,t_a)=E\left[\varphi_0(\eta(t_b))\exp\left\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}V(\eta(u))\,{\rm d}u\right\}\right]$$
$$=\int_{\Omega_-}{\rm d}\mu^\eta(\omega)\varphi_0(\eta(t_b))\exp\{\cdots\} \tag{2}$$

where the path integral measure on the path space $\Omega_-=\{\omega(t):\omega(t_a)=0,\ \eta(t)=Q_a+\omega(t)\}$ given on the manifold $\mathcal{P}$ is defined by the probability distribution of a stochastic process $\eta(t)$. In a local chart $({\rm U},\phi)$ of the manifold $\mathcal{P}$ the process $\eta(t)$ is given by the solution of the stochastic differential equation

$${\rm d}\eta^A(t)=\frac{1}{2}\mu^2\kappa G^{-1/2}\frac{\partial}{\partial Q^B}(G^{1/2}G^{AB})\,{\rm d}t+\mu\sqrt{\kappa}\mathcal{X}^A_{\bar M}(\eta(t))\,{\rm d}w^{\bar M}(t) \tag{3}$$

($\mathcal{X}^A_{\bar M}$ is defined by a local equality $\sum_{\bar K=1}^{n_{\rm P}}\mathcal{X}^A_{\bar K}\mathcal{X}^B_{\bar K}=G^{AB}$, and here and in what follows we denote the Euclidean indices by over-barred indices).

Note that equation (3) is the Stratonovich equation and it transforms in a covariant way under changing the charts of the manifold. It is this defining property that gives one an opportunity to construct a global process on the whole manifold.

We will assume that equation (1) has a fundamental solution $G_P(Q_b,t_b;Q_a,t_a)$, which is defined by the semigroup (2):

$$\psi(Q_a,t_a)=\int G_P(Q_b,t_b;Q_a,t_a)\varphi_0(Q_b)\,{\rm d}v_{\rm P}(Q_b)\equiv(G_P\varphi_0)(Q_a,t_a)$$

(${\rm d}v_{\rm P}(Q)=\sqrt{G(Q)}\,{\rm d}Q^1\cdot\ldots\cdot{\rm d}Q^{n_{\rm P}}$).

If in equation (2) $\varphi_0(Q)=G^{-1/2}(Q)\delta(Q-Q')$ is set, we get the probability representation of the kernel $G_P(Q_b,t_b;Q_a,t_a)$ of the semigroup (2). This can be done by a less formal approach, if we consider the appropriate limit of the approximating functions.

3. Transition to fibre coordinates

Let a smooth free action of a compact Lie group $\mathcal{G}$ be given on a compact manifold $\mathcal{P}$. We assume, in addition, that this right action is isometric and the group $\mathcal{G}$ is unimodular and semisimple. Then, the manifold $\mathcal{P}$ has a fibred structure and there is a principal fibre bundle $\pi:\mathcal{P}\to\mathcal{P}/\mathcal{G}=\mathcal{M}$ [7], where $\mathcal{M}$ is an orbit space of the action of the group $\mathcal{G}$ on $\mathcal{P}$. The principal bundle structure means that there is a corresponding foliation, which in our case is given by the Killing vectors of the Riemannian metric of the manifold $\mathcal{P}$. From this it follows that we can introduce, at least locally, special coordinates (the adapted coordinates) in which the coordinate functions are separated into two sets. The functions of the first set are variable functions under the group action, and those from the second set are the invariant functions.

As it is usually done, we identify the invariant functions with coordinates on a base manifold $\mathcal{M}$ of the fibre bundle $P(\mathcal{M}, \mathcal{G})$, and the variable functions—with the coordinates on a group manifold $\mathcal{G}$ of this fibre bundle.

Hence, we change the coordinates Q^A of the manifold $\mathcal{P}$ for the adapted coordinates (x^i, a^α) consistent with the structure of the fibre bundle $P(\mathcal{M}, \mathcal{G})$.

As a result, the right invariant metric G_{AB} becomes the Kaluza–Klein metric [8]:

$$\begin{pmatrix} h_{ij}(x) + A_i^\mu(x)A_j^\nu(x)\bar{\gamma}_{\mu\nu}(x) & A_i^\mu(x)\bar{u}_\sigma^\nu(a)\bar{\gamma}_{\mu\nu}(x) \\ A_i^\mu(x)\bar{u}_\sigma^\nu(a)\bar{\gamma}_{\mu\nu}(x) & \bar{u}_\rho^\mu(a)\bar{u}_\sigma^\nu(a)\bar{\gamma}_{\mu\nu}(x) \end{pmatrix}. \tag{4}$$

The orbit space metric $h_{ij}(x)$ of (4) is defined as follows [2]: by using the Killing vectors $K_\alpha^A(Q)\frac{\partial}{\partial Q^A}$ and the metric along the orbits $d_{\alpha\beta} = K_\alpha^A G_{AB} K_\beta^B$, we transform G_{AB} into $G_{AB}^\perp = \Pi_A^C G_{CD} \Pi_B^D$ with the help of the projectors $\Pi_B^A = \delta_B^A - K_\alpha^A d^{\alpha\beta} K_{\beta B}$. After transition to the adapted coordinates given by $Q^A = f^A(x^i, a^\alpha)$[1], we obtain the metric $h_{ij}(x)$ from the following formula:

$$h_{ij}(x) = G_{AB}^\perp \frac{\partial f^A}{\partial x^i}\frac{\partial f^B}{\partial x^j}.$$

The mechanical connection $A_i^\mu(x)$ is a pull-back of the Lie algebra-valued connection 1-form $\Omega = \Omega^\alpha \otimes e_\alpha$ by the preferred section. In terms of the initial metric G_{AB} the 1-form Ω^α are given as follows [2, 3, 8]:

$$\Omega^\alpha(Q) = d^{\alpha\beta}(Q) G_{AB}(Q) K_\beta^B(Q)\, \mathrm{d}Q^A.$$

Finally, the expression at the bottom right corner of matrix (4) is the metric on the orbit over x. The matrix $\bar{u}_\beta^\alpha(a)$ is an inverse matrix to matrix $\bar{v}_\beta^\alpha(a) = \frac{\partial \Phi^\alpha(b,a)}{\partial b^\beta}|_{b=e}$. Φ is the composition function of the group: for $c = ab$, $c^\alpha = \Phi^\alpha(a, b)$.

In new coordinates the determinant of the metric G_{AB} is equal to

$$\det G_{AB} = (\det h_{ij}(x))\,(\det \bar{\gamma}_{\alpha\beta}(x))\,(\det \bar{u}_\rho^\mu(a))^2.$$

In the path integral of equation (2) a local transition to the adapted coordinates (x^i, a^α), when one neglects the effects coming from the nontrivial topology of the manifold, can be realized by the stochastic process methods.

The transformation of the measure in the path integral is derived from the phase-space transformation of the stochastic process $\eta^A(t) = f^A(x^i(t), a^\alpha(t))$. We change the stochastic process $\eta^A(t)$ for a new process $\zeta^A(t)$ with the coordinates $(x^i(t), a^\alpha(t))$. Since the phase-space transformation of the stochastic process conserves the probabilities, the path integral of equation (2) transforms into the path integral

$$\psi(Q_a, t_a) = E\left[\tilde{\varphi}_0(x^i(t_b), a^\alpha(t_b)) \exp\left\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b} \tilde{V}(x(u))\,\mathrm{d}u\right\}\right] \tag{5}$$

where $\tilde{\varphi}_0(x, a) = \varphi_0(f(x, a))$, $\tilde{V}(x) = V(f(x, a))$ (in case of the invariance of the potential term $V(Q)$)) and the boundary values of $x_a^i \equiv x^i(t_a)$ and $a_a^\alpha \equiv a^\alpha(t_a)$ in the right-hand side of equation (5) should be expressed in terms of Q_a with the help of inverse transformation f^{-1}.

[1] An explicit expression of the function f depends on the choice of an invariant variable x.

The process $\zeta^A(t)$ that generates the measure in the path integral of equation (5) is described by the following local stochastic differential equations:

$$\begin{aligned}
\mathrm{d}x^i(t) &= \frac{1}{2}\mu^2\kappa\left[\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^n}(h^{ni}\sqrt{h\bar{\gamma}})\right]\mathrm{d}t + \mu\sqrt{\kappa}X^i_{\bar{n}}(x(t))\,\mathrm{d}w^{\bar{n}}(t)\\
\mathrm{d}a^\alpha(t) &= \mu^2\kappa\bigg[-\frac{1}{2}\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^k}\left(\sqrt{h\bar{\gamma}}h^{km}A^\nu_m\right)\bar{v}^\alpha_\nu(a(t))\\
&\quad+\frac{1}{2}(\bar{\gamma}^{\lambda\epsilon}+h^{ij}A^\lambda_iA^\epsilon_j)\bar{v}^\sigma_\lambda(a(t))\frac{\partial}{\partial a^\sigma}(\bar{v}^\alpha_\epsilon(a(t))\bigg]\mathrm{d}t\\
&\quad+\mu\sqrt{\kappa}\bar{v}^\alpha_\lambda(a(t))\bar{Y}^\lambda_{\bar{\epsilon}}\,\mathrm{d}w^{\bar{\epsilon}}(t) - \mu\sqrt{\kappa}X^i_{\bar{n}}A^\nu_i\bar{v}^\alpha_\nu(a(t))\,\mathrm{d}w^{\bar{n}}(t).
\end{aligned}\tag{6}$$

Here $\bar{\gamma} = \det\gamma_{\alpha\beta}(x)$, $h = \det h_{ij}(x)$, $X^i_{\bar{n}}$ and $\bar{Y}^\lambda_{\bar{\epsilon}}$ are defined by the local equalities $\sum_{\bar{n}=1}^{n_M} X^i_{\bar{n}}(x)X^j_{\bar{n}}(x) = h^{ij}(x)$ and $\sum_{\bar{\epsilon}=1}^{n_G} \bar{Y}^\alpha_{\bar{\epsilon}}(a)\bar{Y}^\beta_{\bar{\epsilon}}(a) = \bar{\gamma}^{\alpha\beta}(a)$. The equations (6) have been derived with the help of the Itô differentiation formula.

Note that these local stochastic differential equations transform in a covariant way under changing the charts of the manifold. From this it follows that it is possible to construct the global process $\zeta(t)$ on the principal bundle $P(\mathcal{M},\mathcal{G})$, whose local components coincide with the solutions of the stochastic differential equations (6).

The infinitesimal generator of the transformed semigroup (5) is the Laplace–Beltrami operator for metric (4):

$$\begin{aligned}
\frac{1}{2}\mu^2\kappa\bigg\{&\triangle_M(x) + h^{ij}\frac{1}{\sqrt{\bar{\gamma}}}\left(\frac{\partial\sqrt{\bar{\gamma}}}{\partial x^i}\right)\frac{\partial}{\partial x^j} + h^{ij}A^\alpha_iA^\beta_j\bar{L}_\alpha\bar{L}_\beta - 2h^{in}A^\alpha_n\bar{L}_\alpha\frac{\partial}{\partial x^i} - h^{in}\frac{\partial A^\alpha_n}{\partial x^i}\bar{L}_\alpha\\
&-\frac{h^{in}}{\sqrt{h}}\frac{\partial\sqrt{h}}{\partial x^i}A^\alpha_n\bar{L}_\alpha - h^{in}\frac{1}{\sqrt{\bar{\gamma}}}\frac{\partial\sqrt{\bar{\gamma}}}{\partial x^i}A^\alpha_n\bar{L}_\alpha - \frac{\partial h^{in}}{\partial x^i}A^\alpha_n\bar{L}_\alpha + \bar{\gamma}^{\alpha\beta}\bar{L}_\alpha\bar{L}_\beta\bigg\}
\end{aligned}$$

where $\triangle_M$ is the Laplace–Beltrami operator on $\mathcal{M}$, and by $\bar{L}_\alpha$ we denote the right invariant vector field $\bar{L}_\alpha = \bar{v}^\epsilon_\alpha(a)\frac{\partial}{\partial a^\epsilon}$.

4. Factorization of the measure

Now we should solve the main problem—the problem of the factorization of the measure in the path integral of equation (5). First of all, we make use the properties of conditional expectations of the Markov process to rewrite the right-hand side of equation (5) in the form

$$\psi(Q_a,t_a) = E\left[\exp\left\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}\tilde{V}(x(u))\,\mathrm{d}u\right\}E[\tilde{\varphi}_0(x^i(t_b),a^\alpha(t_b))|(\mathcal{F}_x)^{t_b}_{t_a}]\right]$$

where the path integral $E[\ldots|(\mathcal{F}_x)^{t_b}_{t_a}]$ is the conditional expectation of a function $\tilde{\varphi}_0(x^i(t),a^\alpha(t))$ given a sub-σ-algebra generated by the process $x(t)$ $(t \leqslant t_b)$.

Examining the equations (6) we find that these equations are the same as the stochastic differential equations that are used in the nonlinear filtering theory [9, 10]. The parallel with this theory is achieved, if we consider $x^i(t)$ as the observation process and $a^\alpha(t)$ as the signal process.

It is essential for us that in this theory there is a nonlinear filtering equation, which describes the behaviour of the conditional expectation

$$E[\tilde{\varphi}_0(x^i(t),a^\alpha(t))|(\mathcal{F})_x)^t_{t_a}] \equiv \hat{\tilde{\varphi}}_0(x(t)).$$

It will be convenient to take this equation in the form presented in [10]. With account of equations (6), we write it in the following way:

$$\mathrm{d}\hat{\tilde{\varphi}}_0\,(x(t)) = \mu^2\kappa\left[-\frac{1}{2}\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^k}\left(\sqrt{h\bar{\gamma}}h^{km}A^{\mu}_{m}\right)\right]E[\bar{L}_{\mu}\tilde{\varphi}_0(x^i(t),a^{\alpha}(t))|(\mathcal{F}_x)^t_{t_a}]\,\mathrm{d}t$$
$$+\tfrac{1}{2}\mu^2\kappa(\bar{\gamma}^{\mu\nu}+h^{ij}A^{\mu}_{i}A^{\nu}_{j})E[\bar{L}_{\mu}\bar{L}_{\nu}\tilde{\varphi}_0(x^i(t),a^{\alpha}(t))|(\mathcal{F}_x)^t_{t_a}]\,\mathrm{d}t$$
$$-\mu\sqrt{\kappa}A^{\mu}_{k}X^{k}_{\bar{m}}\,E[\bar{L}_{\mu}\tilde{\varphi}_0(x^i(t),a^{\alpha}(t))|(\mathcal{F}_x)^t_{t_a}]\,\mathrm{d}w^{\bar{m}}\,(t). \tag{7}$$

By using the Peter–Weyl theorem, we develop the function $\tilde{\varphi}_0(x^i,a^{\alpha})$ considered as a function on group $\mathcal{G}$ in series:

$$\tilde{\varphi}_0(x^i,a^{\alpha}) = \sum_{\lambda,p,q} c^{\lambda}_{pq}(x^i)D^{\lambda}_{pq}(a^{\alpha})$$

where $D^{\lambda}_{pq}(a^{\alpha})$ are the matrix elements of an irreducible representation T^{λ}:

$$\sum_q D^{\lambda}_{pq}(a)D^{\lambda}_{qs}(b) = D^{\lambda}_{ps}(ab).$$

Then

$$E[\tilde{\varphi}_0(x^i(t),a^{\alpha}(t))|(\mathcal{F})_x)^t_{t_a}] = \sum_{\lambda,p,q} c^{\lambda}_{pq}(x^i(t))\,E[D^{\lambda}_{pq}(a^{\alpha}(t))|(\mathcal{F}_x)^t_{t_a}].$$

In this formula

$$c^{\lambda}_{pq}(x(t)) = d^{\lambda}\int_{\mathcal{G}}\tilde{\varphi}_0(x(t),\theta)\bar{D}^{\lambda}_{pq}(\theta)\,\mathrm{d}\mu\,(\theta)$$

where d^{λ} is a dimension of an irreducible representation and $\mathrm{d}\mu\,(\theta)$ is a normalized ($\int_{\mathcal{G}}\mathrm{d}\mu\,(\theta)=1$) invariant Haar measure on a group $\mathcal{G}$.

After such a transformation we get the following stochastic differential equation for the conditional expectation: $\hat{D}^{\lambda}_{pq}(x^i(t)) \equiv E[D^{\lambda}_{pq}(a^{\alpha}(t))|(\mathcal{F}_x)^t_{t_a}]$:

$$\mathrm{d}\hat{D}^{\lambda}_{pq}\,(x(t)) = \Gamma^{\mu}_{1}\,(J_{\mu})^{\lambda}_{pq'}\hat{D}^{\lambda}_{q'q}(x(t))\,\mathrm{d}t + \Gamma^{\mu\nu}_{2}\,(J_{\mu})^{\lambda}_{pq'}(J_{\nu})^{\lambda}_{q'q''}\hat{D}^{\lambda}_{q''q}(x(t))\,\mathrm{d}t$$
$$-(J_{\mu})^{\lambda}_{pq'}\hat{D}^{\lambda}_{q'q}(x(t))A^{\mu}_{k}(x(t))X^{k}_{\bar{m}}(x(t))\,\mathrm{d}w^{\bar{m}}\,(t) \tag{8}$$

where the summation on all repeated indices except λ is assumed.

In equation (8) $(J_{\mu})^{\lambda}_{pq} \equiv \big(\frac{\partial D^{\lambda}_{pq}(a)}{\partial a^{\mu}}\big)|_{a=e}$ are infinitesimal generators of the representation $D^{\lambda}(a)$. The coefficients $\Gamma^{\mu}_{1}(x(t))$ and $\Gamma^{\mu\nu}_{2}(x(t))$ are easily derived from equation (7), but for brevity we do not write them explicitly. Also, in deriving equation (8) from (7), we have used the fact that

$$\bar{L}_{\mu}D^{\lambda}_{pq}(a) = \sum_{q'}(J_{\mu})^{\lambda}_{pq'}D^{\lambda}_{q'q}(a).$$

We remark that $\hat{D}^{\lambda}_{pq}(x(t))$ depends also on initial points $x^i_a = x^i(t_a)$ and $\theta^{\alpha}_a = a^{\alpha}(t_a)$ besides the process $x^i(t)$.

Thus, due to the symmetry of our model we have obtained the linear matrix equation for the conditional expectation $\hat{D}^{\lambda}_{pq}$. Its solution can be presented as follows [11, 12]:

$$\hat{D}^{\lambda}_{pq}(x(t)) = (\overleftarrow{\exp})^{\lambda}_{ps}(x(t),t,t_a)\,E[D^{\lambda}_{sq}(a^{\alpha}(t_a))|(\mathcal{F}_x)^t_{t_a}]$$

where by $\overleftarrow{\exp}$ we denote the multiplicative stochastic integral

$$(\overleftarrow{\exp})^{\lambda}_{ps}(x(t),t,t_a) = \overleftarrow{\exp}\int_{t_a}^{t}\Bigg\{\mu^2\kappa\Bigg[\tfrac{1}{2}\bar{\gamma}^{\mu\nu}(x(u))(J_{\mu})^{\lambda}_{pr}(J_{\nu})^{\lambda}_{rs}$$
$$-\frac{1}{2}\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^k}\left(\sqrt{h\bar{\gamma}}h^{km}A^{\mu}_{m}\right)(J_{\mu})^{\lambda}_{ps}\Bigg]\mathrm{d}u$$
$$-\mu\sqrt{\kappa}A^{\mu}_{k}(x(u))(J_{\mu})^{\lambda}_{ps}X^{k}_{\bar{m}}(x(u))\,\mathrm{d}w^{\bar{m}}\,(u)\Bigg\} \tag{9}$$

(h, $\bar{\gamma}$ depend on $x(u)$).

In equation (9) the multiplicative stochastic integral $\overleftarrow{\exp}$ is defined as a limit of the sequence of time-ordered multipliers that have been obtained as a result of breaking the time interval (t, t_a). The arrow aimed to the multipliers given at greater times denotes the time-order of these multipliers.

Notice also, that at the boundary we have

$$E[D^{\lambda}_{sq}(a^{\alpha}(t_a))|(\mathcal{F}_x)^t_{t_a}] = D^{\lambda}_{sq}(a^{\alpha}(t_a)) = D^{\lambda}_{sq}(\theta^{\alpha}_a).$$

As a result of our transformations we get the following relation between the expectation values (the path integrals):

$$(G_P\varphi_0)(Q_a, t_a) = \sum_{\lambda,p,q,q'} E\Bigg[\exp\left\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}\tilde{V}(x(u))\,\mathrm{d}u\right\}c^{\lambda}_{pq}(x(t_b)) \times(\overleftarrow{\exp})^{\lambda}_{pq'}(x(t), t_b, t_a)D^{\lambda}_{q'q}(\theta_a)\Bigg] \tag{10}$$

in which Q_a should be written in terms of (x_a, θ_a) and $(\overleftarrow{\exp})^{\lambda}_{pq'}(\cdots)$ is as in equation (9).

By using the representation of the solution of equation (8) through the multiplicative stochastic integral (9) we will have

$$\frac{1}{2}\mu^2\kappa\Bigg\{\left[\Delta_M + h^{ni}\frac{1}{\sqrt{\bar{\gamma}}}\frac{\partial\sqrt{\bar{\gamma}}}{\partial x^n}\frac{\partial}{\partial x^i}\right](I^{\lambda})_{pq} - 2h^{ni}A^{\alpha}_n(J_{\alpha})^{\lambda}_{pq}\frac{\partial}{\partial x^i} - \frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^n}\left(\sqrt{h\bar{\gamma}}h^{nm}A^{\alpha}_m\right)(J_{\alpha})^{\lambda}_{pq} + (\bar{\gamma}^{\alpha\nu} + h^{ij}A^{\alpha}_i A^{\nu}_j)(J_{\alpha})^{\lambda}_{pq'}(J_{\nu})^{\lambda}_{q'q}\Bigg\}$$

as the infinitesimal generator of the semigroup under the sum in equation (10). In this formula $(I^{\lambda})_{pq}$ is a unity matrix.

It is possible to inverse the equality (10), that is to express the path integral of the right-hand side of equation (10) through our initial path integral. We will do this for the path integral representations of the corresponding Green functions. For this purpose we take the expansion of the delta-function instead of φ_0 in equation (10). Then to perform an inversion of equation (10) we will multiply both sides of it by $\bar{D}(\theta_a)$ and $D(\theta_b)$ and integrate over the boundary group variables with respect to the invariant normalized Haar measure. After that we get

$$G^{\lambda}_{pq}(x_b, t_b; x_a, t_a) = \int_{\mathcal{G}} G_{\mathrm{P}}(\sigma(x_b)\theta, t_b; \sigma(x_a), t_a)D^{\lambda}_{qp}(\theta)\,\mathrm{d}\mu\,(\theta) \tag{11}$$

with the symbolic representation of the Green function G^{λ}_{pq} as

$$G^{\lambda}_{pq}(x_b, t_b; x_a, t_a) = E_{\substack{x(t_a)=x_a\\x(t_b)=x_b}}\left[(\overleftarrow{\exp})^{\lambda}_{pq}(x(t), t_b, t_a)\exp\left\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}\tilde{V}(x(u))\,\mathrm{d}u\right\}\right]$$
$$= \int\limits_{\substack{x(t_a)=x_a\\x(t_b)=x_b}}\mathrm{d}\mu^x\,\exp\left\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}\tilde{V}(x(u))\,\mathrm{d}u\right\}\overleftarrow{\exp}\int_{t_a}^{t_b}\Bigg\{\mu^2\kappa\Bigg[\frac{1}{2}\bar{\gamma}^{\mu\nu}(J_{\mu})^{\lambda}_{pr}(J_{\nu})^{\lambda}_{rq} - \frac{1}{2}\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^k}\left(\sqrt{h\bar{\gamma}}h^{km}A^{\mu}_m\right)(J_{\mu})^{\lambda}_{pq}\Bigg]\mathrm{d}u - \mu\sqrt{\kappa}A^{\mu}_k(J_{\mu})^{\lambda}_{pq}X^k_{\bar{m}}\,\mathrm{d}w^{\bar{m}}\Bigg\}.$$

In deriving relation (11) we have used the property of the right invariance of the Green function G_P and we have presented the coordinates Q^A in terms of x^i and θ^{α}, $Q^A = \sigma^A(x)\theta$, with the help of the local sections $\sigma^A(x) = f^A(x, e)$.

The matrix Green function G^{λ}_{pq} acts in the space of the section of the associated bundle $\mathcal{E} = P \times_{\mathcal{G}} V_{\lambda}$ with the scalar product

$$(\psi_1, \psi_2) = \int_{\mathcal{M}} \langle \psi_1, \psi_2 \rangle_{V_\lambda} \sqrt{\bar{\gamma}(x)}\, \mathrm{d}v_M(x)$$

($\mathrm{d}v_M(x) = \sqrt{h(x)}\, \mathrm{d}x^1 \ldots \mathrm{d}x^{n_M}$, $n_M = \dim \mathcal{M}$ and $\langle \cdot, \cdot \rangle_{V_\lambda}$ is an internal scalar product), provided that we identify the diffusion in x^i—variables with the diffusion on the base space $\mathcal{M}$. The latter can be done by using the method of proof from [13], where the Dynkin theorem on a phase-space transformation of stochastic processes was generalized to be applied to the case of projections of the invariant diffusions.

5. Zero-momentum level reduction

In this section we consider a particular case of formula (11), when $\lambda = 0$. The reduction of this case corresponds to reduction onto a zero-momentum level in the constrained dynamical systems. And as a result of the path integral reduction procedure we will have the integral relation between the path integrals that represent the scalar Green functions.

Since in this case the multiplicative stochastic integral becomes the unity matrix, then the path integral measure of the path integral on the manifold M is now defined by the stochastic process $x^i(t)$:

$$\mathrm{d}x^i(t) = \frac{1}{2}\mu^2\kappa \left[\frac{h^{ni}}{\sqrt{\bar{\gamma}}} \frac{\partial \sqrt{\bar{\gamma}}}{\partial x^n} + \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^n} \left(h^{ni} \sqrt{h} \right) \right] \mathrm{d}t + \mu\sqrt{\kappa} X^i_{\bar{n}}(x(t))\, \mathrm{d}w^{\bar{n}}(t).$$

It follows that the infinitesimal generator of the process $x^i(t)$ is a sum of the Laplace–Beltrami operator and the term which is linear in the partial derivative of x. The standard procedure of the path integral transformation, the Girsanov–Cameron–Martin transformation, allow us to get rid of this additional term. By this procedure, we change the stochastic process $x^i(t)$ for the process $\tilde{x}^i(t)$, whose stochastic differential equation is

$$\mathrm{d}\tilde{x}^i(t) = \frac{1}{2}\mu^2\kappa \left[\frac{1}{\sqrt{h}} \frac{\partial}{\partial x^n} \left(h^{ni} \sqrt{h} \right) \right] \mathrm{d}t + \mu\sqrt{\kappa} X^i_{\bar{n}}(\tilde{x}(t))\, \mathrm{d}w^{\bar{n}}(t).$$

The transformation of the path integral measure is given by

$$\ln \frac{\mathrm{d}\mu^x}{\mathrm{d}\mu^{\tilde{x}}}(\tilde{x}(t)) = \frac{1}{2}\mu\sqrt{\kappa} \int_{t_a}^{t} \frac{1}{\sqrt{\bar{\gamma}}} \frac{\partial \sqrt{\bar{\gamma}}}{\partial x^n} X^n_{\bar{m}}\, \mathrm{d}w^{\bar{m}}(t) - \frac{1}{4}\mu^2\kappa \int_{t_a}^{t} \frac{h^{ni}}{\bar{\gamma}} \frac{\partial \sqrt{\bar{\gamma}}}{\partial x^n} \frac{\partial \sqrt{\bar{\gamma}}}{\partial x^i}\, \mathrm{d}t.$$

In this formula the exponential with the stochastic integral can be replaced by the exponentials with the ordinary integrals. This has been done with the help of the Itô's identity from [5, 14]:

$$\begin{aligned} \exp & \left\{ \frac{1}{2}\mu\sqrt{\kappa} \int_{t_a}^{t} \frac{1}{\sqrt{\bar{\gamma}}} \frac{\partial \sqrt{\bar{\gamma}}}{\partial x^n} X^n_{\bar{m}}\, \mathrm{d}w^{\bar{m}}(t) \right\} \\ &= \left(\frac{\bar{\gamma}(\tilde{x}(t))}{\bar{\gamma}(\tilde{x}(t_a))} \right)^{1/4} \exp \left\{ -\frac{\mu^2\kappa}{4} \int_{t_a}^{t} \left[h^{ni} \frac{\partial^2 \left(\ln \sqrt{\bar{\gamma}} \right)}{\partial x^n \partial x^i} \right. \right. \\ &\quad \left. \left. + \frac{1}{\sqrt{h}} \frac{\partial \left(h^{ni} \sqrt{h} \right)}{\partial x^n} \frac{\partial \left(\ln \sqrt{\bar{\gamma}} \right)}{\partial x^i} + \frac{1}{2} \frac{h^{ni}}{\bar{\gamma}} \frac{\partial \sqrt{\bar{\gamma}}}{\partial x^n} \frac{\partial \sqrt{\bar{\gamma}}}{\partial x^i} \right] \mathrm{d}t \right\}. \end{aligned}$$

After these transformations we get the following integral relation:

$$(\bar{\gamma}(x_b)\bar{\gamma}(x_a))^{-1/4}\, G_M(x_b, t_b; x_a, t_a) = \int_{\mathcal{G}} G_P(\sigma(x_b)\theta, t_b; \sigma(x_a), t_a)\, \mathrm{d}\mu(\theta). \tag{12}$$

The Green function G_M determines a semigroup which acts in the Hilbert space with a scalar product $(\psi_1, \psi_2) = \int \psi_1(x)\psi_2(x)\,\mathrm{d}v_M(x)$. The path integral representation of G_M is given by

$$G_M(x_b, t_b; x_a, t_a) = \int \mathrm{d}\mu^{\tilde{x}}(\omega) \exp\left\{\int_{t_a}^{t_b} \left[\frac{1}{\mu^2\kappa m}\tilde{V}(\tilde{x}(u)) + J(\tilde{x}(u))\right]\mathrm{d}u\right\}$$

where an additional potential term, the Jacobian of the quantum reduction, is

$$J(x) = -\frac{\mu^2\kappa}{8}\left[\Delta_M \ln\bar{\gamma} + \frac{1}{4}h^{ni}\frac{\partial \ln\bar{\gamma}}{\partial x^n}\frac{\partial \ln\bar{\gamma}}{\partial x^i}\right].$$

In (x_b, t_b)-variables, the Green function G_M satisfies the forward Kolmogorov equation with the operator

$$\hat{H}_\kappa = \frac{\hbar\kappa}{2m}\Delta_M - \frac{\hbar\kappa}{8m}\left[\Delta_M \ln\bar{\gamma} + \frac{1}{4}(\nabla_{\mathrm{M}} \ln\bar{\gamma})^2\right] + \frac{1}{\hbar\kappa}\tilde{V}.$$

At $\kappa = i$ this forward Kolmogorov equation becomes the Schrödinger equation with the Hamilton operator $\hat{H} = -\frac{\hbar}{\kappa}\hat{H}_\kappa|_{\kappa=i}$.

Thus, the reduction procedure in the Wiener path integrals representing the evolution of finite-dimensional dynamical systems with a symmetry give rise an additional potential term—the reduction Jacobian. It is worth remarking that this potential term, which is usually supposed to come from the ordering procedure in the Hamiltonian operator associated with the reduced classical Hamiltonian, has an interesting representation. It can be written as some differential expression depending on the mean curvature, which is normal to the orbit above the point of the base space [15].

Acknowledgments

I thank A V Razumov for many useful discussions on geometry, V O Soloviev and V I Borodulin for valuable advice and suggestions. I also wish to thank C DeWitt–Morette for helpful discussions of the path integral reduction problems.

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